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TOPICS IN COMPLEX ANALYSIS
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SHEET 10

Solution for 'Topics in complex analysis'

(19/11/2025)

H 10.1 (On Picard's little theorem)

a) Use Proposition 6.12 to prove Picard's little theorem without using Picard's great theorem.

b) Show that Picard's little theorem is equivalent to the following statement: if $f, g : \mathbb{C} \to \mathbb{C}$ are entire functions such that $e^f + e^g = 1$, then f and g are constant.

Solution H 10.1:

a) Let $f : \mathbb{C} \to \mathbb{C}$ be non-constant and holomorphic. Assume that there exist two elements $w_1, w_2 \in \mathbb{C}$ such that $\{w_1, w_2\} \cap f(\mathbb{C}) = \emptyset$. Then the entire function $g(z) = \frac{f(z)-w_1}{w_2-w_1}$ is again non-constant and omits the values $\{0, 1\}$. By Proposition 6.12 we find a holomorphic function $h : \mathbb{C} \to \mathbb{C}$ such that

$$g = \frac{1}{2}(1 + \cos(\pi \cos(\pi h)))$$

and $h(\mathbb{C})$ does not contain any open ball of radius 1. From the general version of Bloch's theorem (Corollary 6.7), we infer that h is constant, which shows that g and therefore also f is constant. This yields a contradiction.

b) We have seen in the proof of a) that Picard's little theorem is equivalent to the statement that every entire function that omits the two values $\{0, 1\}$ is constant.
Assume that this statement is true. If $1 = e^f + e^g$ for some entire functions $f, g : \mathbb{C} \to \mathbb{C}$, then the function e^g omits the values 0 and 1, so that e^g is constant. Taking the derivative we see that $e^g g' = 0$, so that $g' = 0$. Thus g is constant. Repeating the argument we conclude that f is also constant.
Conversely, assume that for all entire functions $f, g : \mathbb{C} \to \mathbb{C}$ the equality $1 = e^f + e^g$ implies that f and g are constant. Suppose that $h : \mathbb{C} \to \mathbb{C}$ is entire and omits the values $\{0, 1\}$. Then $h = e^g$ for some entire function $g : \mathbb{C} \to \mathbb{C}$. Moreover, there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $1 - h = e^f$. Thus $1 = e^f + e^g$, so f and g are constant. This implies that h is constant. □

H 10.2 (Landau's improvement of Picard's little theorem)

a) Show that there exists a function $R : \mathbb{C} \setminus \{0, 1\} \to (0, +\infty)$ such that for any $a \in \mathbb{C} \setminus \{0, 1\}$ we have

$$\left\{ f \in \mathcal{H}(\overline{B_{R(a)}(0)}) : \ f(0) = a, \ f'(0) = 1, f \text{ omits } \{0, 1\} \right\} = \emptyset.$$

Hint: Set $R(a) = 3L(\frac{1}{2}, |a|)$, where L is given by Schottky's theorem.

b) Show that the statement in a) implies Picard's little theorem.

Solution H 10.2:

a) Following the hint, we set $R(a) = 3L(\frac{1}{2}, |a|)$, which is defined for all $a \in \mathbb{C} \setminus \{0\}$. Now fix $a \in \mathbb{C} \setminus \{0, 1\}$ and assume by contradiction that there exists a function $f \in \mathcal{H}(\overline{B_{R(a)}(0)})$ such that $f(0) = a$, $f'(0) = 1$, and f omits the values $\{0, 1\}$. Note that the Taylor series of f is of the form

$$f(z) = a + z + \sum_{k \geq 2} a_k z^k$$

in an open neighborhood of $\overline{B_{R(a)}(0)}$. Consider then the function $g \in \mathcal{H}(\overline{B_1(0)})$ defined by $g(z) = f(R(a)z)$. It satisfies $g(0) = a$ and omits the values 0 and 1, so that Schottky's theorem implies that

$$\sup_{z \in B_{1/2}(0)} |g(z)| \leq L(1/2, |a|) = \frac{1}{3} R(a).$$

However, by Cauchy's formula for derivatives we have

$$R(a) = |g'(0)| = \left| \frac{1}{2\pi i} \int_{\partial B_{1/2}(0)} \frac{g(z)}{z^2}\, dz \right| \leq 2 \sup_{z \in B_{1/2}(0)} |g(z)| \leq \frac{2}{3} R(a).$$

This yields a contradiction.

b) Let $f : \mathbb{C} \to \mathbb{C}$ be entire and non-constant. Assume by contradiction that $f(\mathbb{C}) \cap \{0, 1\} = \emptyset$. Choose $z_0 \in \mathbb{C}$ such that $f'(z_0) \neq 0$ and set $a = f(z_0) \in \mathbb{C} \setminus \{0, 1\}$. Then the function

$$h(z) := f(z_0 + z/f'(z_0))$$

is entire, satisfies $h(0) = a$ and $h'(0) = 1$, but omits the values $\{0, 1\}$. This contradicts a).

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H 10.3 (Equivalent version of Picard's great theorem)
Prove that Picard's great theorem (Theorem 6.2) is equivalent to the following statement: for any holomorphic function $f : B_1(0) \setminus \{0\} \to \mathbb{C} \setminus \{0, 1\}$, either f or $1/f$ is bounded in a neighborhood of 0.

Solution H 10.3:
Assuming Picard's great theorem, any holomorphic function $f : B_1(0) \setminus \{0\} \to \mathbb{C}$ that omits the values $\{0, 1\}$ cannot have an essential singularity at 0. If the singularity is removable, then f is bounded in a neighborhood of 0. Otherwise, f has a pole at 0, so $1/f$ has a removable singularity (in fact a zero) at 0 and we conclude that $1/f$ is bounded in a neighborhood of 0.
In order to prove the converse statement, we argue by contradiction. Assume that there exist a function $f : B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ with an essential singularity at $z_0 \in \mathbb{C}$, and two distinct values $a, b \in \mathbb{C}$ such that on $B_r(z_0) \setminus \{z_0\}$ the function f assumes the values a and b only finitely many times. After shrinking $r > 0$ we can assume that f omits the values a and b, and by considering

$$z \mapsto \frac{f(rz + z_0) - a}{b - a}$$

we may assume that $r = 1$, $z_0 = 0$, $a = 0$, and $b = 1$. Then by assumption either f or $1/f$ is bounded in a neighborhood of 0. If this is the case for f, then f has a removable singularity at 0, which yields a contradiction. Thus $g = 1/f$ has a removable singularity at 0. Denote the holomorphic extension still by g. If $g(0) \neq 0$, then $f = 1/g$ is also bounded near 0 and we get a contradiction as before. If $g(0) = 0$ then $\lim_{z \to 0} |f(z)| = +\infty$ and we conclude that f has a pole at 0. Hence in both cases we obtain a contradiction.

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H 10.4 (An even sharper version of Montel's theorem)
Let $G \subset \mathbb{C}$ be a simply connected domain and for $m \in \mathbb{N}$ define

$$\mathcal{F}_m := \{f : G \to \mathbb{C} \text{ holomorphic with } f(G) \cap \{0\} = \emptyset \text{ and } \#\{f = 1\} \leq m\},$$

where # denotes the cardinality of a set. Show that for any sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}_m$, either the whole sequence $|f_n|$ converges locally uniformly to $+\infty$, or there exists a subsequence $\{f_{n_j}\}_{j \in \mathbb{N}}$

that converges locally uniformly to a holomorphic function $f : G \to \mathbb{C}$.
Hint: Consider a suitable root $\sqrt[k]{f}$ for $f \in \mathcal{F}_m$.

Solution H 10.4:
Since $f(G) \cap \{0\} = \emptyset$ for every $f \in \mathcal{F}_m$ and G is simply connected, we can define the holomorphic function $\sqrt[m+1]{f} : G \to \mathbb{C} \setminus \{0\}$. Note that there exists $s_f \in \{0, \dots, m\}$ such that

$$\sqrt[m+1]{f(z)} \neq \exp\left(\frac{2\pi i s_f}{m+1}\right) \qquad \forall z \in G.$$

Indeed, otherwise the equation $f(z) = 1$ would have at least $m + 1$ distinct solutions, which contradicts the fact that $f \in \mathcal{F}_m$. Set $z_f = \exp\left(\frac{2\pi i s_f}{m+1}\right) \in \partial B_1(0)$ and consider the rescaled functions $\widetilde{f} = \sqrt[m+1]{f}/z_f : G \to \mathbb{C} \setminus \{0, 1\}$. Given a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_m$, we can apply the sharpened version of Montel's theorem (Theorem 6.14) to the sequence $\{\widetilde{f_n}\}_{n\in\mathbb{N}}$ and deduce that either $|\widetilde{f_n}| \to +\infty$ locally uniformly, or that a subsequence $\{\widetilde{f_{n_j}}\}_{j\in\mathbb{N}}$ converges locally uniformly to a holomorphic function $\widetilde{f} : G \to \mathbb{C}$. After passing to a further subsequence, in this second case we can also assume that $z_{f_{n_j}}$ converges to some $z_\infty \in \partial B_1(0)$. This yields the claim once we unfold the definitions.

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